

## Review Paper

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# Clarification about Component Mode Synthesis Methods for Substructures with Physical Flexible Interfaces

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## Abstract

The objective of the paper is to clarify a methodology based on the use of the existing component mode synthesis methods for the case of two damped substructures which are coupled through a linking viscoelastic flexible substructure and for which the structural modes with free geometrical interface are used for each main substructure. The proposed methodology corresponds to a convenient alternative to the direct use either of the Craig-Bampton method applied to the three substructures (using the fixed geometric interface modes) or of the flexibility residual approaches initiated by MacNeal (using the free geometric interface modes). In opposite to a geometrical interface which is a topological interface on which there is a direct linkage between the degrees of freedom of substructures, we consider a physical flexible interface which exists in certain present technologies and for which the general framework linear viscoelasticity is used and yields a frequency-dependent damping and stiffness matrices of the physical flexible interface.

**Key words:** Component mode synthesis, Dynamic substructuring, Flexible interface, Computational mechanics

## Nomenclature

$N$  = number of elastic modes with fixed interfaces  $\Gamma_1 \cup \Gamma_2$  of  $S$   
 $N_r$  = number of structural modes with free interface of  $S_r$   
 $\mathbf{F}$  = vector of given forces in  $S$  associated with  $\mathbf{U}$   
 $\mathbf{F}^r$  = vector of given forces in  $S_r$  associated with  $\mathbf{U}^r$   
 $\mathbf{G}$  = vector of given forces in  $S$  associated with  $\mathbf{V}$   
 $\mathbf{G}^r$  = vector of given forces in  $S_r$  associated with  $\mathbf{V}^r$   
 $\mathbf{H}$  = vector of given forces in  $S$  associated with  $\mathbf{W}$   
 $\mathbf{H}^r$  = vector of given forces in  $S_r$  associated with  $\mathbf{W}^r$   
 $\mathbf{U}$  = vector of the DOFs in  $S$   
 $\mathbf{U}^r$  = vector of the DOFs in  $S_r$   
 $\mathbf{V}$  = vector of the DOFs of  $S$  on  $\Gamma_1 \cup \Gamma_2$

$\mathbf{V}^r$  = vector of the DOFs of  $S_r$  on  $\Gamma_r$   
 $\mathbf{W}$  = vector of the DOFs in  $S$  and not in  $\Gamma_1 \cup \Gamma_2$   
 $\mathbf{W}^r$  = vector of the DOFs in  $S_r$  and not in  $\Gamma_r$   
 $S$  = linking substructure  
 $S^r$  = main substructure ( $r = 1, 2$ )  
 $n$  =  $n_1 + n_2$ , total number of DOFs on geometrical interfaces  $\Gamma_1 \cup \Gamma_2$   
 $n^r$  = number of DOFs on geometrical interface  $\Gamma_r$  of  $S_r$   
 $m$  = total number of DOFs of linking substructure  $S$   
 $m^r$  = total number of DOFs of main substructure  $S_r$   
 $p$  =  $m - n$   
 $p^r$  =  $m_r - n_r$   
 $\omega$  = frequency in rad/s  
 $\Gamma_r$  = geometrical interface between  $S_r$  and  $S$

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- $\Phi_0$  = matrix of the elastic modes with fixed interfaces  $\Gamma_1 \cup \Gamma_2$  of  $S$
- $\Phi^r$  = matrix of the structural modes with free interface of  $S_r$

## 1. Introduction

The present paper deals with a component mode synthesis method adapted to the analysis of a structural configuration made up of two coupled substructures through a physical flexible interface for which an adapted component synthesis method is the Hurty method that the authors have used and which can be viewed as an extension to a dynamical interface of the Kuhar-Stahle paper in 1974 [1] which was devoted to the static flexible coupling interface including a shifting iterative method, this last aspect being not necessary for the methodology proposed. The dynamics of the interface is considered as soon as the thickness of the interface is not too small and the flexibility of the interface is relatively significant. Since such the methodology proposed to analyze such a configuration has never explicitly been detailed in the literature (at the knowledge of the author), the objective of the paper is to propose a clarification about such a formulation based on existing component synthesis methods. The proposed methodology corresponds to a convenient alternative to the direct use either of the Craig-Bampton method applied to the three substructures (using the fixed geometric interface modes) or of the flexibility residual approaches initiated by MacNeal (using the free geometric interface modes).

In this paper, a geometrical interface means a topological interface on which there is a direct linkage between the degrees of freedom of substructures. In opposite, a physical flexible interface must be understood as a third continuum medium constituting a third flexible substructure. Taking into account the nature of the physical flexible interface existing in certain present technologies, the authors also take the opportunity to propose a more general framework for the physical flexible interface which is then assumed to have a linear viscoelastic behavior which yields a frequency-dependent damping and stiffness matrices of the physical flexible interface.

Among all the given references, let us cite two important interesting general reviews which have been published by Craig in 1985 [2] and de Klerk, Rixen and Voormeeren in 2008 [3], in which the component mode synthesis methods and variants are described. Nevertheless, in order to position the methodology proposed with respect to the existing component mode synthesis methods, we give a brief review

of the history of the major developments in this field. The readers are referred to [2, 3] for complete details of these general methods.

We are then interested in the construction of a reduced-order model for linear vibration of a damped structure subjected to prescribed forces and composed of two main linear damped substructures connected through a linking linear viscoelastic flexible substructure.

The two main substructures are thus coupled through a dissipative physical interface. Such a reduced-order model allows the frequency response function calculations to be carried out. More precisely, this paper is devoted to computational aspects of a substructure coupled with another substructure through a third linking substructure, using a dynamic substructuring method and a modal reduction procedure, under the hypothesis that the two main substructures are represented by their structural modes with free geometrical interface (the structural modes are defined as the ensemble of the elastic modes (modes presenting deformations) and the rigid body modes if they exist).

This situation is often encountered due to experimental considerations and/or due to engineering specifications for each main substructures, in particular, in aerospace engineering.

The concept of substructures to perform a matrix structural analysis was first introduced by Argyris and Kelsey in 1959 [4] and by Przemieniecki in 1963 [5] who introduced the decomposition of the static displacement field in structural analysis into two spaces, one is the space of the substructure with fixed boundary and the second one is the correction of boundary relaxation, later called, the static boundary functions in dynamic substructuring. The Przemieniecki work devoted to static structural problem was extended by Guyan and Irons in 1965 [6, 7] and to structural dynamics using as an approximation the static boundary functions for the mass matrix. In 1960 and 1965, Hurty [8, 9] considered the case of two substructures coupled through a geometrical interface, for which the first substructure is represented using its elastic modes with fixed geometrical interface and the second substructure is represented using its elastic modes with free geometrical interface completed by static boundary functions of the first substructure. Finally, Craig and Bampton in 1968 [10] adapted the Hurty method in order to represent each substructure of the same manner consisting in using the elastic modes of the substructure with fixed geometrical interface and the static boundary functions on its geometrical interface. For complex dynamical systems with many appendages considered as substructures (such as disk with blades), Benfield and Hrudá in 1971 [11] proposed a component mode substitution using the Craig and Bampton

method for each appendage.

Starting from these pioneering fundamental works, in the last four decades, improvements of the dynamic substructuring methodology have been proposed with many variants (see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25]).

Another type of methods has been introduced in order to use the structural modes with free geometrical interface for two coupled substructures instead of the structural modes with fixed geometrical interface (elastic modes) as used in the Craig and Bampton method. In this context, MacNeal in 1971 [26] introduced the concept of residual flexibility and then used by Rubin in 1975 [27]. From this pioneering fundamental papers, several works have been published and among them, let us cite [28, 29, 30, 31, 32, 33]. In this context, the Lagrange multipliers have also been used to write the coupling on the geometrical interface [23, 34, 35, 36]

As damping plays an important role in the prediction of the dynamical responses, substructuring techniques taking into account damping have been the subject of several investigations, for instance, [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. In particular, the damping modeling is important in the medium-frequency range which has been analyzed in [48, 49, 50, 51] in the context of substructuring techniques. In addition, uncertainty quantification is nowadays recognized as playing an important role in order to improve the robustness of the numerical simulations for the low-frequency range and especially, in the medium-frequency range and some works have been developed in the context of substructuring techniques [52, 53, 54, 55, 56, 57, 58].

Experimental modal substructuring, which is very useful for updating computational models of substructures, has been developed and several papers devoted to this subject have been published, among them [59, 60, 61, 62, 63, 64, 65, 66].

Finally, some other aspects of substructuring techniques related to sensitivity analysis, response surface, interpolation with respect to system parameters, control and optimization can be found in [67, 68, 69, 70, 71, 72, 73, 74, 75].

Reviews have also been performed [2, 3]. It should be noted that all these above dynamic substructuring methodologies, which have been developed for the discrete case (computational model), have also been reanalyzed in the framework of the continuous case (continuum mechanics) by Morand and Ohayon in 1979 [76] for which details can be found [77] for conservative systems and by Ohayon and Soize in 1998 [42] for the dissipative systems.

General computational methods, computational linear structural dynamics and vibration, algorithms for solving large eigenvalue problems and uncertainty quantification

will not be developed in this paper and we refer the reader to the following basic reference books [78, 79, 80, 81, 82, 83, 84, 85, 86, 77, 42, 87, 88, 89, 90, 91, 92].

A physical interface (the linking substructure) between two coupled substructures, modeled by an elastic medium, has been considered by [1] without introducing additional degrees of freedom in the junction. In this paper, a generalization of the Kuhar and Stahle work [1] is proposed. The theoretical aspects adapted to computational dynamics are thus presented for linear elastodynamic of a damped structure composed of two main damped substructures coupled through a dissipative physical interface made up of a linking damped substructure. A reduced-order model is constructed using the structural modes of the two main substructures with free geometrical interface and, for the linking substructure, using an appropriate vector basis with fixed geometrical interfaces and an appropriate static boundary matrix with respect to the geometrical interfaces.

In order to preserve the readability of the paper, the theory is presented in the context of computational mechanics using the matrix formulation corresponding to the discretized system (finite element discretization) and without giving the continuous formulation that can be found in [93]. As the methodology presented is supported by theoretical arguments, numerical examples are not presented in this paper.

We consider a structure composed of two damped main substructures  $S_1$  and  $S_2$  coupled through a dissipative physical interface  $S$  (linking damped substructure) by two geometrical interfaces  $\Gamma_1$  and  $\Gamma_2$  (see Fig. 1). The matrix equations are written in the frequency domain. The linking damped substructure  $S$  is modeled in the context of the general linear viscoelasticity theory [94, 95], yielding damping and stiffness matrices depending on the frequency, while the two main damped substructures  $S_1$  and  $S_2$  are modeled in the context of linear elasticity with an additional classical damping modeling which is assumed to be independent of the frequency (the extension to the case of frequency-dependent damping matrix is straightforward). The methodology proposed can be summarized as follows.

As explained above, it is assumed that the two main substructures are represented using the structural modes of each main substructure with free geometrical interface. The methodology proposed is based on the use of an adapted combination of the Guyan [6], Hurty [8, 9] and Craig & Bampton [10] works. Firstly, the structural modes (elastic modes presenting deformations and rigid body modes if they exist) are computed for each substructure  $S_r$  with free

geometrical interface  $\Gamma_r$ . Secondly, an appropriate vector basis corresponding to a generalized eigenvalue problem for the linking substructure  $S$  is computed with fixed geometrical interfaces  $\Gamma_1 \cup \Gamma_2$  and the static boundary functions (static boundary matrix corresponding to a Schur complement calculation of the stiffness matrix at zero frequency in the context of matrix analysis) are calculated. In dynamic substructuring, when the structural modes with free geometrical interface are used for the substructures, the common methodology consists in using the residual flexibility methods and/or the Lagrange multipliers procedures. We investigate here the alternative methodology summarized above.

## 2. Structural modes of main substructure $S_r$ with free geometrical interface $\Gamma_r$

As explained above, the problem is formulated in the frequency domain for which the frequency is denoted by  $\omega$  (in rad/s). For  $r=1,2$ , the  $m_r$ -DOFs computational model of main substructure  $S_r$ , with free geometrical interface  $\Gamma_r$ , is represented by the  $(m_r \times m_r)$  real matrices  $\mathbf{M}^r$ ,  $\mathbf{K}^r$  and  $\mathbf{D}^r$ , and by the real vectors  $\mathbf{U}^r$  and  $\mathbf{F}^r$  of dimension  $m_r$ . The mass matrix  $\mathbf{M}^r$  is positive and invertible (positive definite). The damping matrix  $\mathbf{D}^r$  and the stiffness matrix  $\mathbf{K}^r$ , which are assumed to be frequency independent, are positive and, either invertible (positive definite if there are no rigid body motions), or not invertible (positive semidefinite if there are rigid body motions). The frequency-dependent displacements vector  $\mathbf{U}^r = (\mathbf{V}^r, \mathbf{W}^r)$  is decomposed into the vector  $\mathbf{V}^r$  of dimension  $n_r$  (number of DOFs on geometrical interface  $\Gamma_r$ ) and the vector  $\mathbf{W}^r$  of dimension  $p_r = m_r - n_r$  (number of the other DOFs). The frequency-dependent vector of external given forces  $\mathbf{F}^r = (\mathbf{G}^r, \mathbf{H}^r)$  is decomposed into the vector  $\mathbf{G}^r$  of dimension  $n_r$  (components of the forces relative to the  $m_r$  DOFs of the geometrical interface  $\Gamma_r$ ) and the vector  $\mathbf{H}^r$  of dimension  $p_r = m_r - n_r$  (components of the forces relative to the other DOFs).

The  $N_r \leq m_r$  structural modes of substructure  $S_r$  with

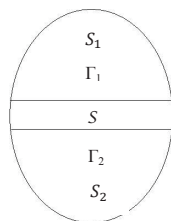


Fig. 1. Two substructures  $S_1$  and  $S_2$  connected with a linking structure  $S$

free geometrical interface  $\Gamma_r$  are calculated with the computational model solving the classical generalized eigenvalue problem,

$$\mathbf{K}^r \Phi^r = \mathbf{M}^r \Phi^r \Lambda^r, \quad (1)$$

in which  $\Lambda^r$  is the diagonal matrix of the eigenvalues  $\lambda_1^r \leq \dots \leq \lambda_{N_r}^r$  and where  $\Phi^r$  is the rectangular  $(m_r \times N_r)$  real matrix whose columns are the structural modes associated with eigenfrequencies  $\Omega_1^r \leq \dots \leq \Omega_{N_r}^r$ , in which  $\Omega_\alpha^r = \sqrt{\lambda_\alpha^r}$ . If rigid body motions exist, the corresponding zero eigenvalues and the associated structural modes are included in the sequence above. Vector  $\mathbf{U}^r$  is projected on the structural modes and writes

$$\mathbf{U}^r = \Phi^r \mathbf{q}^r, \quad (2)$$

in which  $\mathbf{q}^r$  is the vector of dimension  $N_r$  of the generalized coordinates. With respect to the decomposition  $\mathbf{U}^r = (\mathbf{V}^r, \mathbf{W}^r)$ , Eq. (2) yields

$$\mathbf{V}^r = \Phi_V^r \mathbf{q}^r, \quad \mathbf{W}^r = \Phi_W^r \mathbf{q}^r, \quad (3)$$

In which matrix  $\Phi^r$  has been written in the block form with respect to  $\mathbf{U}^r = (\mathbf{V}^r, \mathbf{W}^r)$  as

$$\Phi^r = \begin{bmatrix} \Phi_V^r \\ \Phi_W^r \end{bmatrix}. \quad (4)$$

## 3. Reduced-order matrix model of linking flexible viscoelastic substructure $S$

Similarly to the previous matrix description of the two main substructures, the  $m$ -DOFs computational model of linking substructure  $S$ , with free geometrical interfaces  $\Gamma_1$  and  $\Gamma_2$ , is represented by the  $(m \times m)$  real matrices  $\mathbf{M}$ ,  $\mathbf{D}(\omega)$  and  $\mathbf{K}(\omega)$ , and by the real vectors  $\mathbf{U}$  and  $\mathbf{F}$  of dimension  $m$ . The mass matrix  $\mathbf{M}$  is positive and invertible. The damping matrix  $\mathbf{D}(\omega)$  and the stiffness matrix  $\mathbf{K}(\omega)$ , which depend on the frequency, are positive and, either invertible (if there are no rigid body motions), or not invertible (if there are rigid body motions). The frequency-dependent displacements vector  $\mathbf{U} = (\mathbf{V}, \mathbf{W})$  is decomposed into the vector  $\mathbf{V}$  of dimension  $n = n_1 + n_2$  (number of DOFs on geometrical interface  $\Gamma = \Gamma_1 \cup \Gamma_2$ ) and the vector  $\mathbf{W}$  of dimension  $p = m - n$  (number of the other DOFs). The frequency-dependent vector of external given forces  $\mathbf{F} = (\mathbf{G}, \mathbf{H})$  is decomposed into the vector  $\mathbf{G}$  of dimension  $n$  (components of the forces relative to the  $m$  DOFs of the geometrical interface  $\Gamma$ ) and the vector  $\mathbf{H}$  of dimension  $p = m - n$  (components of the forces relative to the other DOFs).

### 3.1. Vector basis for linking substructure $S$ with fixed geometrical interface $\Gamma = \Gamma_1 \cup \Gamma_2$

Let  $\mathbf{M}_0$  and  $\mathbf{K}_0(\omega)$  be the  $(p \times p)$  mass and stiffness matrices of the linking substructure  $S$  with fixed geometrical interface  $\Gamma = \Gamma_1 \cup \Gamma_2$  (deduced from  $\mathbf{M}$  and  $\mathbf{K}(\omega)$ ). For fixed  $\omega$ , the  $N \leq p$  eigenvectors of linking substructure  $S$  with fixed geometrical interface  $\Gamma$  are then computed solving the generalized eigenvalue problem depending on  $\omega$  considered as a parameter,

$$\mathbf{K}_0(\omega) \Phi_0(\omega) = \mathbf{M}_0 \Phi_0(\omega) \Lambda(\omega) \quad (5)$$

in which  $\Lambda(\omega)$  is the diagonal matrix of the eigenvalues  $0 < \lambda_1(\omega) \leq \dots \leq \lambda_N(\omega)$  and where  $\Phi_0(\omega)$  is the rectangular  $(p \times N)$  real matrix whose columns are the eigenvectors  $\phi_0^1(\omega), \dots, \phi_0^N(\omega)$  associated with the eigenvalues  $0 < \lambda_1(\omega) \leq \dots \leq \lambda_N(\omega)$ . It should be noted, that for all fixed  $\omega$  considered as a parameter, the family  $\{\phi_0^1(\omega), \dots, \phi_0^p(\omega)\}$  is a vector basis of  $\mathbb{R}^p$  and  $\{\phi_0^1(\omega), \dots, \phi_0^N(\omega)\}$  spans a subspace of  $\mathbb{R}^p$  of dimension  $N$

**Remarks.** The reduced-order model of order  $N$ , which is constructed for analyzing the response of the structure for  $\omega$  belonging to a given frequency band of analysis  $B = [\omega_{\min}, \omega_{\max}]$  with  $0 \leq \omega_{\min} < \omega_{\max}$ , is chosen independent of  $\omega$ . In practice, the response is calculated for the frequencies belonging to the set  $\mathcal{B} = \{\omega_1, \omega_2, \dots, \omega_\mu\}$  of  $\mu$  sampling frequencies of band  $B$ . In practice  $\mu$  can be of several hundreds or of the order of one thousand.

In the above formulation, Eq. (5) must be solved for all  $\omega$  in  $\mathcal{B}$ . If the number  $p$  of DOFS and the number  $\mu$  of frequencies is not too high (that is generally the case for the linking substructure and a standard frequency analysis), such a computation remains feasible. It should be noted that massively parallel computers facilitate the analysis of such parameterized numerical problem.

2) For solving Eq. (5), the numerical cost can be reduced using the following procedure. First, Eq. (5) is solved for  $\omega$  belonging to the set  $\mathcal{B}' = \{\omega'_1, \dots, \omega'_{\mu'}\}$  of master frequency points in  $\mathcal{B}$  with  $\mu' \ll \mu$ . Then,  $\Phi_0(\omega)$  is calculated for all  $\omega$  in  $\mathcal{B}'$  by using an interpolation procedure based on the values  $\Phi_0(\omega'_1), \dots, \Phi_0(\omega'_{\mu'})$ , at the  $\mu'$  master frequency points. Finally, the reduced generalized eigenvalue problem  $\Phi_0(\omega)^T \mathbf{K}_0(\omega) \Phi_0(\omega) = \Phi_0(\omega)^T \mathbf{M}_0(\omega) \tilde{\Lambda}(\omega)$  of small dimension  $N$  is solved for all  $\omega$  in  $\mathcal{B}$  in order to construct the corresponding approximation  $\tilde{\Lambda}(\omega)$  of  $\Lambda(\omega)$  for the  $\mu$  frequencies in  $\mathcal{B}$ . Such a procedure can be found in [73].

3) Another way consists in replacing the above interpolation procedure by the following construction of a frequency-independent basis adapted to band  $\mathcal{B}$ . It consists in extracting the larger family of linearly independent vectors from the family  $\{\Phi_0(\omega'_1), \dots, \Phi_0(\omega'_{\mu'})\}$

4) If  $\mathbf{K}_0(\omega)$  slowly varies on band  $B$ , a well adapted frequency-independent basis for all  $\omega$  in  $B$ , consists in choosing  $\Phi_0(\omega)$  for a given  $\omega$  in  $B$  but in such a case convergence with respect to  $N$  must be carefully checked.

5) More generally, any subset of  $N$  vectors extracted from a family of linearly independent vectors can be used. Let us cite, for instance, the methods belonging to the class of the Proper Orthogonal Decomposition (POD) procedure.

### 3.2. Static boundary functions for linking flexible substructure $S$ with respect to the geometrical interface $\Gamma = \Gamma_1 \cup \Gamma_2$

The static boundary matrix is constructed for  $\omega=0$ . The block decomposition of matrix  $\mathbf{K}(0)$  with respect to  $\mathbf{U}=(\mathbf{V}, \mathbf{W})$  is written as

$$\mathbf{K}(0) = \begin{bmatrix} \mathbf{K}_V(0) & \mathbf{K}_{VW}(0) \\ \mathbf{K}_{VW}(0)^T & \mathbf{K}_0(0) \end{bmatrix}. \quad (6)$$

The static boundary matrix is then defined as the  $(p \times n)$  real matrix  $\mathbf{S}$  such that

$$\mathbf{S} = -\mathbf{K}_0(0)^{-1} \mathbf{K}_{VW}(0)^T. \quad (7)$$

### 3.3. Projection matrix for the construction of reduced-order model of linking flexible viscoelastic substructure $S$

For all  $\omega$  fixed in  $\mathcal{B}$ , we introduce the projection matrix  $\mathbf{T}_s(\omega)$  which is written by blocks with respect to  $\mathbf{U}=(\mathbf{V}, \mathbf{W})$  as

$$\mathbf{T}_s(\omega) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{S} & \Phi_0(\omega) \end{bmatrix} \quad (8)$$

in which  $\mathbf{I}_n$  is the  $(n \times n)$  identity matrix. Vector  $\mathbf{U}=(\mathbf{V}, \mathbf{W})$  is then the transformation of vector  $(\mathbf{V}, \mathbf{q})$  such that  $\mathbf{W} = \mathbf{S}\mathbf{V} + \Phi_0(\omega)\mathbf{q}$  in which  $\mathbf{q}$  is the vector of the generalized coordinates of dimension  $N$ . The kinematic coupling conditions on geometrical interfaces  $\Gamma_1$  and  $\Gamma_2$  writes  $\mathbf{V} = (\mathbf{V}^1, \mathbf{V}^2)$  and the previous equation is then rewritten as

$$\mathbf{W} = \mathbf{R}^1 \mathbf{q}^1 + \mathbf{R}^2 \mathbf{q}^2 + \Phi_0(\omega)\mathbf{q}, \quad (9)$$

in which, for  $r=1, 2$ , the frequency-independent  $(p \times N_r)$  real matrix  $\mathbf{R}^r$  is written by blocks,

$$\mathbf{R}^r = \begin{bmatrix} \mathbf{S}_{1r} \Phi_V^r \\ \mathbf{S}_{2r} \Phi_V^r \end{bmatrix}, \quad (10)$$

in which the following block decomposition of matrix  $\mathbf{S}$ , relative to  $(\mathbf{V}^1, \mathbf{V}^2)$ , has been used,

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \quad (11)$$

#### 4. Reduced-order matrix model of the assembled structure $S_1 \cup S \cup S_2$

##### 4.1. Projection matrix for constructing the reduced-order matrix model of the assembled structure

For the assembled structure  $S_1 \cup S \cup S_2$ , using Eq. (2) for  $r=1$  and  $r=2$  and using Eq. (9) yield the projection matrix  $\mathbf{T}(\omega)$  which relates the physical displacement vector  $(\mathbf{U}^1, \mathbf{W}, \mathbf{U}^2)$  of dimension  $m_1+p+m_2$  to the generalized coordinates  $(\mathbf{q}^1, \mathbf{q}, \mathbf{q}^2)$ , such that

$$\begin{bmatrix} \mathbf{U}^1 \\ \mathbf{W} \\ \mathbf{U}^2 \end{bmatrix} = \begin{bmatrix} \Phi^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{R}^1 & \Phi_0(\omega) & \mathbf{R}^2 \\ \mathbf{0} & \mathbf{0} & \Phi^2 \end{bmatrix} \begin{bmatrix} \mathbf{q}^1 \\ \mathbf{q} \\ \mathbf{q}^2 \end{bmatrix}. \quad (12)$$

It should be noted that the viscoelasticity behavior of the physical flexible inter-face implies that the projection matrix defined by Eq. (12) depends on  $\omega$ .

##### 4.2. Dynamic stiffness matrix and given forces vector for each substructure

Below,  $\omega$  is the frequency considered as a real parameter. For  $r=1, 2$ , the dynamic stiffness matrix of the main substructure  $S_r$  is the symmetric  $(m_r \times m_r)$  complex matrix  $\mathbf{A}^r(\omega)$  which is written as  $\mathbf{A}^r(\omega) = -\omega^2 \mathbf{M}^r + i\omega \mathbf{D}^r + \mathbf{K}^r$ . The dynamic stiffness matrix of the linking flexible viscoelastic substructure  $S$  is the symmetric  $(m \times m)$  complex matrix  $\mathbf{A}(\omega)$  which is written as  $\mathbf{A}(\omega) = -\omega^2 \mathbf{M} + i\omega \mathbf{D} + \mathbf{K}$ . The following block decomposition of matrix  $\mathbf{A}(\omega)$ , with respect to  $(\mathbf{V}^1, \mathbf{V}^2, \mathbf{W})$  of dimension  $n_1+n_2+p$ , is defined as

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{A}_{V_1}(\omega) & \mathbf{0} & \mathbf{A}_{V_1 W}(\omega) \\ \mathbf{0} & \mathbf{A}_{V_2}(\omega) & \mathbf{A}_{V_2 W}(\omega) \\ \mathbf{A}_{V_1 W}(\omega)^T & \mathbf{A}_{V_2 W}(\omega)^T & \mathbf{A}_0(\omega) \end{bmatrix}, \quad (13)$$

and the given frequency-dependent forces vector  $\mathbf{F} = (\mathbf{G}, \mathbf{H})$  is rewritten as  $\mathbf{F} = (\mathbf{G}_{V_1}, \mathbf{G}_{V_2}, \mathbf{H})$  of dimension  $n_1+n_2+p$ . For  $r=1, 2$ , the  $(m_r \times m_r)$  complex matrix  $\mathbf{B}_r(\omega)$ , the  $(m_r \times m_r)$  complex matrix  $\mathbf{C}_r(\omega)$  and the frequency-dependent  $(m_r \times 1)$  complex matrix  $\mathbf{J}^r$  are defined by blocks as follows

$$\mathbf{B}_r(\omega) = \begin{bmatrix} \mathbf{A}_{V_r}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{C}_r(\omega) = \begin{bmatrix} \mathbf{A}_{V_r W}(\omega) \\ \mathbf{0} \end{bmatrix}, \mathbf{J}^r = \begin{bmatrix} \mathbf{G}_{V_r} \\ \mathbf{0} \end{bmatrix}. \quad (14)$$

Finally, we introduce the symmetric  $(m_1+p+m_2) \times (m_1+p+m_2)$  complex matrices,  $\mathbf{A}^1(\omega)$ ,  $\mathbf{A}^2(\omega)$  and  $\mathbf{A}(\omega)$  defined by blocks, with respect to  $(\mathbf{U}^1, \mathbf{W}, \mathbf{U}^2)$ , by

$$\mathbf{A}^1(\omega) = \begin{bmatrix} \mathbf{A}^1(\omega) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{A}^2(\omega) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^2(\omega) \end{bmatrix} \quad (15)$$

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{B}_1(\omega) & \mathbf{C}_1(\omega) & \mathbf{0} \\ \mathbf{C}_1(\omega)^T & \mathbf{A}_0(\omega) & \mathbf{C}_2(\omega)^T \\ \mathbf{0} & \mathbf{C}_2(\omega) & \mathbf{B}_2(\omega) \end{bmatrix}, \quad (16)$$

and the frequency-dependent  $m_1+p+m_2$  complex vectors  $\mathbf{F}^1(\omega)$ ,  $\mathbf{F}^2(\omega)$  and  $\mathbf{F}(\omega)$  defined by blocks, with respect to  $(\mathbf{U}^1, \mathbf{W}, \mathbf{U}^2)$ , by

$$\mathbb{F}^1 = \begin{bmatrix} \mathbf{F}^1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \mathbb{F}^2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}^2 \end{bmatrix}, \mathbb{F} = \begin{bmatrix} \mathbf{J}_{V_1} \\ \mathbf{H} \\ \mathbf{J}_{V_2} \end{bmatrix} \quad (17)$$

Consequently, the equilibrium dynamic equation for the  $m_1+p+m_2$  complex displacements vector,  $\mathbf{U} = (\mathbf{U}^1, \mathbf{W}, \mathbf{U}^2)$ , of the assembled structure  $S_1 \cup S \cup S_2$ , is written as

$$[\mathbf{A}^1(\omega) + \mathbf{A}(\omega) + \mathbf{A}^2(\omega)] \mathbf{U} = \mathbf{F}^1 + \mathbf{F} + \mathbf{F}^2. \quad (18)$$

##### 4.3. Reduced-order matrix model of the assembled structure

For all frequency  $\omega$  considered as a parameter, the projection of Eq. (18) is performed using the transformation  $\mathbf{T}(\omega)$  defined by Eq. (12). For the assembled structure, the reduced-order matrix model is obtained in terms of the vector  $\mathbf{q} = (\mathbf{q}^1, \mathbf{q}, \mathbf{q}^2)$  of dimension  $N_1+N+N_2$  and is such that  $\mathbf{U} = \mathbf{T}(\omega) \mathbf{q}$ , in which  $\mathbf{q}$  is solution of the following symmetric reduced complex matrix equation,

$$[\mathcal{A}^1(\omega) + \mathcal{A}(\omega) + \mathcal{A}^2(\omega)] \mathbf{q} = \mathcal{F}^1 + \mathcal{F} + \mathcal{F}^2, \quad (19)$$

in which  $\mathcal{A}^r(\omega) = \mathbf{T}(\omega)^T \mathbf{A}^r(\omega) \mathbf{T}(\omega)$ ,

$\mathcal{A}(\omega) = \mathbf{T}(\omega)^T \mathbf{A}(\omega) \mathbf{T}(\omega)$ ,  $\mathcal{F} = \mathbf{T}(\omega)^T \mathbf{F}$  and  $\mathcal{F} = \mathbf{T}(\omega)^T \mathbf{F}$ . For all  $\omega$  belonging to frequency band of analysis  $B = [\omega_{\min}, \omega_{\max}]$ , the convergence of the reduced-order matrix model has to be performed with respect to the three parameters  $N_1, N$  and  $N_2$ .

## 5. Conclusion

We have clarified an alternative methodology based on the use of the existing component mode synthesis methods for the case of two damped substructures coupled through a linking viscoelastic flexible substructure and for which the structural modes with free geometrical interface are used for each main substructure. The notion of physical flexible interface has also been defined. More generally, this model can be adapted for complex linking flexible substructure including smart materials, semi-active or active controls. In general for a complex linking substructure, there are model uncertainties induced by modeling errors which can be taken into account in the context of the methodology proposed.

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